

# MEAN VALUE RESULTS AND $\Omega$ -RESULTS FOR THE HYPERBOLIC LATTICE POINT PROBLEM IN CONJUGACY CLASSES

DIMITRIOS CHATZAKOS

**ABSTRACT.** For  $\Gamma$  a Fuchsian group of finite covolume, we study the lattice point problem in conjugacy classes on the Riemann surface  $\Gamma \backslash \mathbb{H}$ . Let  $\mathcal{H}$  be a hyperbolic conjugacy class in  $\Gamma$  and  $\ell$  the  $\mathcal{H}$ -invariant closed geodesic on the surface. The main asymptotic for the counting function of the orbit  $\mathcal{H} \cdot z$  inside a circle of radius  $t$  centered at  $z$  grows like  $c_{\mathcal{H}} \cdot e^{t/2}$ . This problem is also related with counting distances of the orbit of  $z$  from the geodesic  $\ell$ . For  $X \sim e^{t/2}$  we study mean value and  $\Omega$ -results for the error term  $e(\mathcal{H}, X; z)$  of the counting function. We prove that a normalized version of the error  $e(\mathcal{H}, X; z)$  has finite mean value in the parameter  $t$ . Further, we prove that if  $\Gamma$  is cocompact then

$$\int_{\ell} e(\mathcal{H}, X; z) ds(z) = \Omega \left( X^{1/2} \log \log \log X \right).$$

For  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$  we prove the same  $\Omega$ -result, using a subconvexity bound for the Epstein zeta function associated to an indefinite quadratic form in four variables. We also study pointwise  $\Omega_{\pm}$ -results for the error term. Our results extend the work of Phillips and Rudnick for the classical lattice problem to the conjugacy class problem.

## 1. INTRODUCTION

**1.1. Mean value and  $\Omega$ -results for the classical hyperbolic lattice point problem.** Let  $\mathbb{H}$  be the hyperbolic plane,  $z, w$  two fixed points in  $\mathbb{H}$  and  $\rho(z, w)$  their hyperbolic distance. For  $\Gamma$  a cocompact or cofinite Fuchsian group, the classical hyperbolic lattice point problem asks to estimate the quantity

$$N(X; z, w) = \# \left\{ \gamma \in \Gamma : \rho(z, \gamma w) \leq \cosh^{-1} \left( \frac{X}{2} \right) \right\},$$

as  $X \rightarrow \infty$ . This problem has been studied by many authors [1, 5, 6, 8, 10, 11, 19, 20, 22]. One of the main methods to understand this problem is using the spectral theory of automorphic forms. For this reason, let  $\Delta$  be the Laplacian of the hyperbolic surface  $\Gamma \backslash \mathbb{H}$  and let  $\{u_j\}_{j=0}^{\infty}$  be the  $L^2$ -normalized eigenfunctions (Maass forms) of  $-\Delta$  with eigenvalues  $\{\lambda_j\}_{j=0}^{\infty}$ . We also write  $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2$ . Selberg [22], Günther [10], Good [8] et. al. proved that

$$(1.1) \quad N(X; z, w) = \sum_{1/2 < s_j \leq 1} \sqrt{\pi} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z) \overline{u_j(w)} X^{s_j} + E(X; z, w),$$

where the error term  $E(X; z, w)$  satisfies the bound

$$E(X; z, w) = O(X^{2/3}).$$

Conjecturally, the optimal upper bound for the error term  $E(X; z, w)$  is

$$(1.2) \quad E(X; z, w) = O_{\epsilon}(X^{1/2+\epsilon})$$

for every  $\epsilon > 0$  (see [19], [20]). This error term has a spectral expansion over all  $\lambda_j \geq 1/4$ . The contribution of  $\lambda_j = 1/4$  is well understood. We subtract it from  $E(X; z, w)$  and we define the refined

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error term  $e(X; z, w)$  to be the difference

$$e(X; z, w) = E(X; z, w) - h(0) \sum_{t_j=0} u_j(z) \overline{u_j(w)} = E(X; z, w) + O(X^{1/2} \log X),$$

where  $h(t)$  is the Selberg/Harish-Chandra transform of the characteristic function  $\chi_{[0, (X-2)/4]}$  (see [2, p. 2] for the details). Thus, bound (1.2) is equivalent with the bound

$$(1.3) \quad e(X; z, w) = O_\epsilon(X^{1/2+\epsilon})$$

for every  $\epsilon > 0$ . For  $z = w$ , Phillips and Rudnick proved mean value results and  $\Omega$ -results (i.e. lower bounds for the  $\limsup |e(X; z, z)|$ ) that support conjecture (1.3). For  $\Gamma$  cofinite but not cocompact, let  $E_{\mathfrak{a}}(z, s)$  be the nonholomorphic Eisenstein series corresponding to the cusp  $\mathfrak{a}$ . Phillips and Rudnick [20] proved the following theorems.

**Theorem 1.1** (Phillips-Rudnick [20]). *(a) Let  $\Gamma$  be a cocompact group. Then:*

$$(1.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{e(2 \cosh r; z, z)}{e^{r/2}} dr = 0.$$

*(b) Let  $\Gamma$  be a cofinite group. Then:*

$$(1.5) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{e(2 \cosh r; z, z)}{e^{r/2}} dr = \sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2.$$

**Theorem 1.2** (Phillips-Rudnick [20]). *(a) If  $\Gamma$  is cocompact or a subgroup of finite index in  $PSL_2(\mathbb{Z})$ , then for all  $\delta > 0$ ,*

$$e(X; z, z) = \Omega\left(X^{1/2}(\log \log X)^{1/4-\delta}\right).$$

*(b) If  $\Gamma$  is cofinite but not cocompact, and either has some eigenvalues  $\lambda_j > 1/4$  or some cusp  $\mathfrak{a}$  with  $E_{\mathfrak{a}}(z, 1/2) \neq 0$ , then,*

$$e(X; z, z) = \Omega\left(X^{1/2}\right).$$

*(c) If any other cofinite case, for all  $\delta > 0$ ,*

$$e(X; z, z) = \Omega\left(X^{1/2-\delta}\right).$$

In the proof of Theorem 1.2, the assumption  $z = w$  is essential. In [2], we studied  $\Omega$ -results for the average

$$(1.6) \quad M(X; z, w) = \frac{1}{X} \int_2^X \frac{e(x; z, w)}{x^{1/2}} dx$$

for two different points  $z, w$ . We proved that, if  $\lambda_1 > 2.7823\dots$  and  $z, w$  are sufficiently close to each other, the limit of  $M(X; z, w)$  as  $X \rightarrow \infty$  does not exist. In many cases, these results imply pointwise  $\Omega$ -results for  $e(X; z, w)$  with  $z \neq w$  as immediate corollaries.

There are specific groups  $\Gamma$  for which we can provide refined  $\Omega$ -results. In [2], we proved that if  $\Gamma$  is a cofinite group with sufficiently many cusp forms at the point  $z$  in the sense that the series

$$\sum_{|t_j| < T} |u_j(z)|^2 \gg T^2$$

and satisfies  $E_{\mathfrak{a}}(z, 1/2) \neq 0$  for some cusp  $\mathfrak{a}$  then

$$e(X; z, w) = \Omega_{\pm}(X^{1/2})$$

for  $z$  fixed and  $w$  sufficiently close to  $z$  (see [2, corollary 1.9]).

**1.2. The conjugacy class problem.** In this paper we are interested in studying mean value results and  $\Omega$ -results for the hyperbolic lattice point problem in conjugacy classes. In this problem we restrict the action of  $\Gamma$  in a hyperbolic conjugacy class  $\mathcal{H} \subset \Gamma$ ; that means  $\mathcal{H}$  is the conjugacy class of a hyperbolic element of  $\Gamma$ . Let  $z \in \mathbb{H}$  be a fixed point. The problem asks to estimate the asymptotic behavior of the quantity

$$N_z(t) = \#\{\gamma \in \mathcal{H} : \rho(z, \gamma z) \leq t\},$$

as  $t \rightarrow \infty$ . This problem was first studied by Huber in [12, 13]. The main reason we are interested in this problem is because it is related with counting distances of points in the orbit of the fixed point  $z$  from a closed geodesic. This geometric interpretation was first explained by Huber in [12]. Assume  $\mathcal{H}$  is the conjugacy class of the hyperbolic element  $g^\nu$  with  $g$  primitive and  $\nu \in \mathbb{N}$ . Let also  $\ell$  be the invariant closed geodesic of  $g$  (hence  $\ell$  is  $\mathcal{H}$ -invariant). Then  $N_z(t)$  counts the number of  $\gamma \in \Gamma/\langle g \rangle$  such that  $\rho(\gamma z, \ell) \leq t$ . Equivalently,  $N_z(t)$  counts the number of geodesic segments on  $\Gamma \backslash \mathbb{H}$  from  $z$  perpendicular to  $\ell$  of length less than or equal to  $t$ .

Let  $\mu = \mu(\ell)$  be the length of  $\ell$  and let  $X$  be given by the change of variable

$$(1.7) \quad X = \frac{\sinh(t/2)}{\sinh(\mu/2)} \sim c_{\mathcal{H}} \cdot e^{t/2}.$$

Under this parametrization denote  $N_z(t)$  by  $N(\mathcal{H}, X; z)$ . Thus we have

$$N(\mathcal{H}, X; z) = \#\left\{\gamma \in \mathcal{H} : \frac{\sinh(\rho(z, \gamma z)/2)}{\sinh(\mu/2)} \leq X\right\}.$$

The conjugacy class problem holds also a main formula similar to formula (1.1), which can be proved using the spectral theorem for  $L^2(\Gamma \backslash \mathbb{H})$ . This formula was first derived by Good in [8]; it can also be written in the following explicit form, see [4].

**Theorem 1.3** (Good [8], Chatzidakis-Petridis [4]). *Let  $\Gamma$  be a cocompact or cofinite Fuchsian group and  $\mathcal{H}$  a hyperbolic conjugacy class of  $\Gamma$ . Then:*

$$N(\mathcal{H}, X; z) = \sum_{1/2 < s_j \leq 1} A(s_j) \hat{u}_j u_j(z) X^{s_j} + E(\mathcal{H}, X; z),$$

where  $A(s)$  is the product:

$$(1.8) \quad A(s) = 2^{s-1} \left( e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)} \right) \frac{\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \Gamma\left(s - \frac{1}{2}\right)}{\pi \Gamma(s+1)},$$

$$(1.9) \quad \hat{u}_j = \int_{\sigma} \bar{u}_j ds$$

is the period integral of  $\bar{u}_j$  along a segment  $\sigma$  of the invariant closed geodesic of  $\mathcal{H}$  with length  $\int_{\sigma} ds = \frac{\mu}{\nu}$  and

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

Notice that Theorem 1.3 implies the main asymptotic of  $N(\mathcal{H}, X; z)$  is

$$N(\mathcal{H}, X; z) \sim \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \frac{\mu}{\nu} X.$$

Once again we are interested in the growth of the error term. The similarities that arise between the two problems suggest that we should expect the bound

$$(1.10) \quad E(\mathcal{H}, X; z) = O_{\epsilon}(X^{1/2+\epsilon}).$$

(see [4, Conjecture 5.7]). As in the classical problem, the error term  $E(\mathcal{H}, X; z)$  has a ‘spectral expansion’ over the eigenvalues  $\lambda_j \geq 1/4$ . We subtract the contribution of the eigenvalue  $\lambda_j = 1/4$  and we denote the expansion over the eigenvalues  $\lambda_j > 1/4$  by  $e(\mathcal{H}, X; z)$  (eq. (2.13)). In section 2 we prove that the bound (1.10) is equivalent with the bound

$$(1.11) \quad e(\mathcal{H}, X; z) = O_{\epsilon}(X^{1/2+\epsilon}).$$

In order to state our first result, we will need the following definition.

**Definition 1.4.** The Eisenstein period associated to the hyperbolic conjugacy class  $\mathcal{H}$  is the period integral

$$(1.12) \quad \hat{E}_{\mathfrak{a}}(1/2 + it) = \int_{\sigma} E_{\mathfrak{a}}(z, 1/2 - it) ds(z),$$

across a segment  $\sigma$  of the invariant geodesic  $\ell$  with length  $\int_{\sigma} ds = \mu/\nu$ .

In section 3 we prove that the error term  $e(\mathcal{H}, X; z)$  has finite mean value in the radial parameter  $t$ .

**Theorem 1.5.** *Let  $\Gamma$  be a cocompact or cofinite Fuchsian group and for  $x \geq 1$  let  $r$  be defined as  $r = \log(x + \sqrt{x^2 - 1})$ .*

(a) *If  $\Gamma$  is cocompact, then*

$$(1.13) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dr = 0.$$

(b) *If  $\Gamma$  is cofinite, then*

$$(1.14) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dr = \frac{|\Gamma(3/4)|^2}{\pi^{3/2}} \sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2).$$

**Remark 1.6.** Using the change of variables (1.7) we see that Theorem 1.5 is indeed a mean value result in the radial parameter  $t \sim 2r + \mu - 2 \log 2$ . (where the parameter  $t$  counts the distance between the closed geodesic of  $\mathcal{H}$  and the orbit of  $z$ ).

For the conjugacy class problem, proving pointwise  $\Omega$ -results is a more subtle problem comparing to the classical one, due to the appearance of the period integrals in the spectral expansion of  $e(\mathcal{H}, X; z)$ . In the proof of Theorem 1.2, Phillips and Rudnick choose  $z = w$  so that the series expansion of the error term  $e(X; z, w)$  contains the expressions  $|u_j(z)|^2$  which are nonnegative. In this setting, the natural choice is to average over the  $\mathcal{H}$ -invariant geodesic  $\ell$ . For this reason, we will need the following result of Good and Tsuzuki which describes the exact asymptotic behaviour of the period integrals.

**Theorem 1.7** (Good [8], Tsuzuki [23]). *The period integrals  $\hat{u}_j$  of Maaß forms and  $\hat{E}_{\mathfrak{a}}(1/2 + it)$  of Eisenstein series satisfy the asymptotic*

$$\sum_{|t_j| < T} |\hat{u}_j|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^T |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 dt \sim \frac{\mu(\ell)}{\pi} \cdot T,$$

where  $\mu(\ell)$  denotes the length of the invariant closed geodesic  $\ell$ .

We refer to [17, p. 3-4] for a detailed history of this result. We also give the following definition which is related to Theorem 1.7.

**Definition 1.8.** Fix  $\mathcal{H}$  be a hyperbolic class of a cofinite but not cocompact group  $\Gamma$ . We say that the group  $\Gamma$  has sufficiently small Eisenstein periods associated to  $\mathcal{H}$  if for all cusps  $\mathfrak{a}$  we have

$$\int_{-T}^T |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}}$$

for a fixed  $\delta > 0$ .

For the rest of this paper we write  $\int_{\mathcal{H}} ds$  to indicate that we average over a segment of the invariant geodesic  $\ell$  of length  $\mu/\nu$ . When  $\mathcal{H}$  is the class of a primitive element we get  $\nu = 1$ , hence  $\int_{\mathcal{H}} ds = \int_{\ell} ds$ .

We distinguish the two cases of  $\Omega$ -results: if  $g(X)$  is a positive function, we write  $e(X; z, w) = \Omega_+(g(X))$  if

$$\limsup \frac{e(X; z, w)}{g(X)} > 0,$$

and  $e(X; z, w) = \Omega_-(g(X))$  if

$$\liminf \frac{e(X; z, w)}{g(X)} < 0.$$

In section 4 we prove the following theorem, which is an average  $\Omega$ -result on the closed geodesic of  $\mathcal{H}$ .

**Theorem 1.9.** (a) If  $\Gamma$  is either (i) cocompact or (ii) cofinite but not cocompact and has sufficiently small Eisenstein periods associated to  $\mathcal{H}$  according to Definition 1.8, then

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2} \log \log \log X).$$

(b) If  $\Gamma$  is cofinite but not cocompact and either (i)  $\hat{u}_j \neq 0$  for at least one  $\lambda_j > 1/4$  or (ii)  $\hat{E}_{\mathbf{a}}(1/2) \neq 0$  for a cusp  $\mathbf{a}$  then

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2}).$$

**Remark 1.10.** In subsection 4.3 we will see that the modular group  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$  has sufficiently small Eisenstein periods associated to a fixed conjugacy class  $\mathcal{H} \subset \Gamma$ . This follows from a subconvexity bound on the critical line for an Epstein zeta function associated to  $\mathcal{H}$ .

The asymptotic behaviour for the sums of period integrals in Theorem 1.7 is  $c \cdot T$ , where in local Weyl's law (Theorem 2.4) we get an asymptotic  $c \cdot T^2$ . If  $\Gamma$  is cocompact or cofinite but it has sufficiently small Eisenstein periods associated to  $\mathcal{H}$  then

$$(1.15) \quad \sum_{|t_j| < T} |\hat{u}_j|^2 \sim \frac{\mu(\ell)}{\pi} \cdot T,$$

and summation by parts implies

$$(1.16) \quad \sum_{|t_j| < T} \frac{|\hat{u}_j|^2}{t_j} \gg \log T.$$

In case (a) of Theorem 1.9 the triple logarithm should be compared with the extra factor  $(\log \log X)^{1/4-\delta}$  in case (a) of Theorem 1.2. The first is a consequence of the asymptotic behaviour of period integrals in Theorem 1.7, and the second is a consequence of the local Weyl's law.

To prove pointwise  $\Omega$ -results for  $e(\mathcal{H}, X; z)$  we would like to have a fixed pair  $(z, \mathcal{H})$  with  $e(\mathcal{H}, X; z)$  large, i.e. a pair  $(z, \mathcal{H})$  with a uniform 'fixed sign' property of all  $\hat{u}_j u_j(z)$ . That would allow us to prove a pointwise  $\Omega$ -result of the form

$$\limsup_X \frac{|e(\mathcal{H}, X; z)|}{X^{1/2}} = \infty.$$

However, Maass forms have complicated behaviour on the surfaces  $\Gamma \backslash \mathbb{H}$ ; for instance, the nodal domains have very complicated shapes. For this reason we have not been able to determine any such specific pair  $(z, \mathcal{H})$  with the desired fixed sign property. To overcome this problem we notice that the period integral is the limit of Riemann sums. Starting with a fixed conjugacy class  $\mathcal{H}$ , a discrete average allows us to prove the existence of at least one point  $z = z_{\mathcal{H}}$  for which the error  $e(\mathcal{H}, X; z_{\mathcal{H}})$  cannot be small.

We first prove the following proposition for discrete averages.

**Proposition 1.11.** *Let  $\mathcal{H}$  be a fixed hyperbolic class in  $\Gamma$ . If  $\Gamma$  is either (i) cocompact or (ii) if  $\Gamma$  is as in part (b) of Theorem 1.9, then there exist an integer  $K = K_{\mathcal{H}}$  depending only on  $\mathcal{H}$  and  $z_1, z_2, \dots, z_K$  points on  $\ell$  such that:*

$$\frac{1}{K} \sum_{m=1}^K e(\mathcal{H}, X; z_m) = \Omega_+(X^{1/2}).$$

In comparison with our results in [2], in order to prove  $\Omega_-$ -results for the error  $e(\mathcal{H}, X; z)$  we are lead to investigate the behaviour of a modification of the average error term

$$\frac{1}{X} \int_1^X \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dx$$

on the geodesic  $\ell$ .

**Proposition 1.12.** *Let  $\Gamma$  be either (i) cocompact or (ii) cofinite but not cocompact,  $\hat{u}_j \neq 0$  for at least one  $\lambda_j > 1/4$  and  $\hat{E}_{\mathfrak{a}}(1/2) = 0$  for all cusps  $\mathfrak{a}$ . For  $Y = X + \sqrt{X^2 - 1}$  and  $y = x + \sqrt{x^2 - 1}$  let*

$$M_{\mathcal{H}, z}(X) = \frac{1}{Y} \int_1^Y \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dy.$$

*Then there exist an integer  $K = K_{\mathcal{H}}$  and  $z_1, z_2, \dots, z_K$  points in  $\ell$  such that, as  $X \rightarrow \infty$ :*

$$\frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X) = \Omega_-(1).$$

We deduce the following theorem on pointwise  $\Omega$ -results for the error term  $e(\mathcal{H}, X; z)$  as an immediate corollary of Theorem 1.5 and Propositions 1.11, 1.12.

**Theorem 1.13.** *Let  $\Gamma$  be a Fuchsian group,  $\mathcal{H}$  a hyperbolic conjugacy class of  $\Gamma$  and  $\ell$  the invariant closed geodesic of  $\mathcal{H}$ .*

(a) *If  $\Gamma$  is as in Proposition 1.11, then there exist at least one point  $z_{\mathcal{H}} \in \ell$  such that:*

$$e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega_+(X^{1/2}).$$

(b) *If  $\Gamma$  is as in Proposition 1.12, then there exists at least one point  $z_{\mathcal{H}} \in \ell$  such that:*

$$e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega_-(X^{1/2}).$$

(c) *If  $\Gamma$  is not cocompact and the sum  $\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2)$  does not vanish then:*

$$e(\mathcal{H}, X; z) = \Omega(X^{1/2}).$$

Finally, at the last section, as an application of Theorem 1.7 we obtain upper bounds for the error terms of both the classical problem and the conjugacy class problem on geodesics.

**Remark 1.14.** For the proof of Theorem 1.9 and Propositions 1.11, 1.12 we will crucially need some ‘fixed-sign’ properties of the  $\Gamma$ -function stated in Lemma 2.3. We emphasize that the differences in the signs in the two cases of Lemma 2.3 causes the different signs of our  $\Omega$ -results.

**Remark 1.15.** It follows from Theorem 1.13 that in order to prove a pointwise result  $e(\mathcal{H}, X; z) = \Omega(X^{1/2})$  for one point  $z$ , we must only assume the nonvanishing of one period  $\hat{u}_j$ . In this case, the sign of our  $\Omega$ -result can be determined by the vanishing or not of the Eisenstein period integrals. If  $\Gamma$  is cocompact or all Eisenstein periods vanish then there exists at least two points  $z, w \in \ell$  such that:

$$e(\mathcal{H}, X; z) = \Omega_+(X^{1/2}), \quad e(\mathcal{H}, X; w) = \Omega_-(X^{1/2}).$$

These Eisenstein periods are of particular arithmetic interest; in fact  $\hat{E}_{\mathfrak{a}}(1/2)$  is the constant term of the hyperbolic Fourier expansion of  $E_{\mathfrak{a}}(z, s)$  (see [7, section 3.2]). In the arithmetic case, these periods are associated to special values of Epstein zeta functions (see subsection 4.3). We notice that, in principle, it is easier to check the nonvanishing of one period  $\hat{E}_{\mathfrak{a}}(1/2)$  than the nonvanishing of the sum  $\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2)$ .

**Remark 1.16.** Phillips and Rudnick in [20] generalized Theorem 1.1 and case c) of Theorem 1.2 in the case of the  $n$ -dimensional hyperbolic space  $\mathbb{H}^n$  [20, p. 106].

Recently, Paarkonen and Paulin [18] studied the hyperbolic lattice point problem in conjugacy classes for the  $n$ -th hyperbolic space and in a more general setting. However, their geometric approach cannot be used to generalise our results in dimensions  $n \geq 3$ . To do this, we need an explicit expression for the Huber transform  $d_n(f, t)$  in the  $n$ -th dimension. In dimension  $n = 3$ ,  $d_3(f, t)$  was recently studied explicitly by Laaksonen in [16], where he obtained upper bounds for the second moments of the error term, generalising previous work by the author and Petridis [4].

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## 2. SPECTRAL THEORY AND COUNTING

**2.1. The Huber transform.** We briefly state the basic results from the spectral theory of automorphic forms for the conjugacy class problem (see [4, section 2] for the details). Let  $C_0^*[1, \infty)$  denote the space of real functions of compact support that are bounded in  $[1, \infty)$  and have at most finitely many discontinuities.

**Definition 2.1.** Let  $f \in C_0^*[1, \infty)$ . The Huber transform  $d(f, t)$  of  $f$  at the spectral parameter  $t$  is defined as

$$(2.1) \quad d(f, t) = \int_0^{\frac{\pi}{2}} f\left(\frac{1}{\cos^2 v}\right) \frac{\xi_\lambda(v)}{\cos^2 v} dv,$$

with  $\lambda = 1/4 + t^2$ , and  $\xi_\lambda$  is the solution of the differential equation

$$(2.2) \quad \xi_\lambda''(v) + \frac{\lambda}{\cos^2 v} \xi_\lambda(v) = 0, \quad v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

with  $\xi_\lambda(0) = 1$ ,  $\xi_\lambda'(0) = 0$ .

The Huber transform plays a role analogous to that of the Selberg/Harish-Chandra transform in the classical counting (see [4], [13]). For this reason we work with  $d(f, t)$  for an appropriate test function  $f = f_X$ .

**2.2. The test function and counting.** Assume first that  $\Gamma \backslash \mathbb{H}$  is compact. For an  $f \in C_0^*[1, \infty)$  we define the  $\Gamma$ -automorphic function

$$(2.3) \quad A(f)(z) = \sum_{\gamma \in \mathcal{H}} f\left(\frac{\cosh \rho(z, \gamma z) - 1}{\cosh \mu(\gamma) - 1}\right).$$

The function  $A(f)(z)$  has a Fourier expansion of the form

$$(2.4) \quad A(f)(z) = \sum_j 2d(f, t_j) \hat{u}_j u_j(z),$$

where  $d(f, t)$  is the Huber transform of  $f$ . The quantity  $N(\mathcal{H}, X; z)$  can be interpreted as

$$(2.5) \quad A(f_X)(z) = N(\mathcal{H}, X; z),$$

for  $f_X = \chi_{[1, X^2]}$ , the characteristic function of the interval  $[1, X^2]$ . We have the following lemma.

**Lemma 2.2.** *Let  $s = 1/2 + it$ . For the Huber transform of  $f_X$  we have the following estimates.*

(a) *If  $s \in (1/2, 1]$  then*

$$2d(f_X, t) = A(s)X^s + O\left(\Gamma(s - 1/2)X^{s-2} + \Gamma(1/2 - s)X^{1-s}\right),$$

where  $A(s)$  is the  $\Gamma$ -product defined in (1.8).

(b) For  $t \in \mathbb{R} - \{0\}$  and  $U = \sqrt{X^2 - 1}$  we have

$$\begin{aligned} 2d(f_X, t) &= \frac{\sqrt{2}}{\pi} \left| \Gamma\left(\frac{3}{4} + \frac{it}{2}\right) \right|^2 \Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)} \left(e^{-\frac{i\pi}{4} - \frac{\pi t}{2}} + e^{\frac{i\pi}{4} + \frac{\pi t}{2}}\right) (X + U)^{it}\right) X^{1/2} \\ &\quad + O\left((1 + |t|)^{-2} X^{-3/2} (X + U)^{it}\right) \end{aligned}$$

(c) For  $t = 0$  we have

$$d(f_X, 0) = O(X^{1/2} \log X).$$

Before giving the proof of the Lemma, for the rest of the paper we fix the notation  $U = \sqrt{X^2 - 1}$ ,  $Y = X + U$ ,  $R = \log Y = \log(X + U)$ ,  $y = x + \sqrt{x^2 - 1}$  and  $r = \log(x + \sqrt{x^2 - 1})$ . We also fix the notation

$$\begin{aligned} D(t) &= \frac{\sqrt{2}}{\pi} \left| \Gamma\left(\frac{3}{4} + \frac{it}{2}\right) \right|^2, \\ a(t) &= e^{-\frac{i\pi}{4} - \frac{\pi t}{2}} + e^{\frac{i\pi}{4} + \frac{\pi t}{2}}, \\ G(t) &= \frac{D(t)a(t)}{\Gamma(3/2 + it)}. \end{aligned} \tag{2.6}$$

Stirling's formula implies that, as  $|t| \rightarrow \infty$ ,

$$D(t) \sim 4e^{-\frac{\pi|t|}{2}} (1 + |t|)^{1/2}, \quad \frac{|\Gamma(it)|}{|\Gamma(3/2 + it)|} \sim (1 + |t|)^{-3/2}. \tag{2.7}$$

Therefore, we have the asymptotic

$$|G(t)\Gamma(it)| \asymp (1 + |t|)^{-1}. \tag{2.8}$$

We now give the proof of the Lemma 2.2.

*Proof.* (a) Using the integral representation for  $d(f_X, t)$  in [4, p. 5] we get

$$d(f_X, t) = (2\sqrt{\pi})^{-1} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \int_0^U (P_{s-1}^0(iv) + P_{s-1}^0(-iv)) dv. \tag{2.9}$$

Using [9, p. 968, eq. (8.752.3)], this takes the form

$$d(f_X, t) = (2\sqrt{\pi})^{-1} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) X (P_{s-1}^{-1}(iU) - P_{s-1}^{-1}(-iU)). \tag{2.10}$$

Using formula [9, p. 971, eq. (8.776)], the statement follows.

(b) We use [9, p. 971, eq. (8.774)], so that equation (2.10) gives

$$2d(f_X, t) = \Re\left(G(t)\Gamma(it)(X + U)^{it} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{X - U}{2X}\right)\right) X^{1/2}, \tag{2.11}$$

where  $F(a, b; c; z)$  denotes the Gauss' hypergeometric function. As  $X \rightarrow \infty$ , the definition of the hypergeometric function [9, p. 1005, eq. (9.100)] implies

$$F\left(-\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{X - U}{2X}\right) = 1 + O((1 + |t|)^{-1} X^{-2}). \tag{2.12}$$

We finish the proof of part b) using the asymptotics (2.7).

(c) It follows after elementary calculations, using eq. (2.10), formula [9, p. 961, eq. (8.713.2)] and estimates.  $\square$



Using (a) of Lemma 2.2 and [4, prop. 2.4] we obtain that, in the compact case, the error term  $E(\mathcal{H}, X; z)$  has a ‘spectral expansion’ of the form

$$E(\mathcal{H}, X; z) = \sum_{t_j \in \mathbb{R}} 2d(f_X, t_j) \hat{u}_j u_j(z) + O \left( \sum_{1/2 < s_j \leq 1} \Gamma(s_j - 1/2) X^{s_j - 2} + \Gamma(1/2 - s_j) X^{1 - s_j} \right).$$

The  $s_j$ ’s are discrete, thus we can find a constant  $\sigma = \sigma_\Gamma \in (0, 1/2]$  such that  $s_j - 1/2 \geq \sigma$  for all  $s_j \in (1/2, 1]$ . This implies that the above  $O$ -term is  $O(X^{1/2 - \sigma})$ . Using (c) of Lemma 2.2 and the finiteness of the eigenspace for the eigenvalue  $t_j = 0$  we get the bound

$$d(f_X, 0) \sum_{t_j = 0} \hat{u}_j u_j(z) = O(X^{1/2} \log X).$$

As in the classical problem, we extract this quantity from  $E(\mathcal{H}, X; z)$  and we define the error term  $e(\mathcal{H}, X; z)$  to be the difference

$$(2.13) \quad e(\mathcal{H}, X; z) = E(\mathcal{H}, X; z) - d(f_X, 0) \sum_{t_j = 0} \hat{u}_j u_j(z).$$

Hence, for  $\Gamma$  cocompact we conclude the principal series of the error  $e(\mathcal{H}, X; z)$  takes the form:

$$(2.14) \quad e(\mathcal{H}, X; z) = \sum_{t_j > 0} 2d(f_X, t_j) \hat{u}_j u_j(z) + O(X^{1/2 - \sigma}).$$

**2.3. Some more auxiliary lemmas.** One of the key ingredients in the proofs of our results is the following lemma.

**Lemma 2.3.** *For every  $t \in \mathbb{R} - \{0\}$ , we have:*

a)

$$\Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} a(t) \right) > 0,$$

b)

$$\Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)} a(t) \right) < 0.$$

Lemma 2.3 can be proved elementary, as [2, Lemma 2.1], using properties of Beta function  $B(x, y)$  and integration by parts. It can also be verified using Mathematica.

We will finally need the following estimate for the Maass forms and the Eisenstein series which is a local version of Weyl’s law for  $L^2(\Gamma \backslash \mathbb{H})$ .

**Theorem 2.4** (Local Weyl’s law). *For every  $z$ , as  $T \rightarrow \infty$ ,*

$$\sum_{|t_j| < T} |u_j(z)|^2 + \sum_a \frac{1}{4\pi} \int_{-T}^T |E_a(z, 1/2 + it)|^2 dt \sim cT^2,$$

where  $c = c(z)$  depends only on the number of elements of  $\Gamma$  fixing  $z$ .

See [20, p. 86, lemma 2.3] for a proof of this result. We emphasize that if  $z$  remains in a compact set of  $\mathbb{H}$  the constant  $c(z)$  remains uniformly bounded.

## 3. THE MEAN VALUE RESULT

**3.1. Proof of Theorem 1.5 for  $\Gamma \backslash \mathbb{H}$  compact.** We first prove the error term  $e(\mathcal{H}, X; z)$  has zero mean value for  $\Gamma$  cocompact.

*Proof.* In this case  $\Gamma$  has only discrete spectrum. The characteristic function  $f_X$  is not smooth, hence we cannot apply the spectral theorem for  $L^2(\Gamma \backslash \mathbb{H})$  [14, p. 69, Theorem 4.7 and p. 103, Theorem 7.3] directly to  $A(f_X)$ , as the principal series in (2.14) diverges. Instead, we will work with the average

$$M_{\mathcal{H}}(T) = \frac{1}{T} \int_0^T \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dr.$$

Using (2.14) and part (b) of Lemma 2.2 we conclude

$$\begin{aligned} M_{\mathcal{H}}(T) &= \sum_{t_j > 0} \Re \left( G(t_j) \Gamma(it_j) \frac{1}{T} \int_0^T e^{it_j r} dr \right) \hat{u}_j u_j(z) \\ &\quad + O \left( \sum_{t_j > 0} |t_j|^{-2} \hat{u}_j u_j(z) \frac{1}{T} \int_0^T e^{(-2+it_j)r} dr + \frac{1}{T} \int_0^T e^{-r\sigma} dr \right). \end{aligned}$$

Using Theorems 1.7, 2.4 and Stirling's formula (estimate (2.7)) we bound the main term and the  $O$ -terms of  $M_{\mathcal{H}}(T)$  as

$$M_{\mathcal{H}}(T) \ll \frac{1}{T} \sum_{t_j > 0} |t_j|^{-2} \hat{u}_j u_j(z) + O \left( \frac{1}{T} \sum_{t_j > 0} |t_j|^{-3} \hat{u}_j u_j(z) + \frac{1}{T} \int_0^T e^{-r\sigma} dr \right) = O(T^{-1}),$$

and the statement follows.  $\square$

**3.2. Proof of Theorem 1.5 for  $\Gamma$  for cofinite.** In this case the hyperbolic Laplacian  $-\Delta$  has also continuous spectrum which is spanned by the Eisenstein series  $E_{\mathfrak{a}}(z, 1/2 + it)$  (see [14, chapters 3,6 and 7]). To prove case (b) of Theorem 1.5 we have to consider the contribution of the continuous spectrum in  $M_{\mathcal{H}}(T)$ , which is given in terms of the Eisenstein series  $E_{\mathfrak{a}}(z, 1/2 + it)$  and the period integrals  $\hat{E}_{\mathfrak{a}}(1/2 + it)$ . More specifically, using [4, eq. (4.1)] and [4, Lemma 4.2] we get that the contribution of the continuous spectrum is given by

$$(3.1) \quad \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z, 1/2 + it) \left( \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr \right) dt.$$

To complete the proof of Theorem 1.5, we need to prove that the expansion in (3.1) converges to

$$\frac{|\Gamma(3/4)|^2}{\pi^{3/2}} \sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2)$$

as  $T \rightarrow \infty$ . To deal with this expansion, we need the following lemma for the Huber transform.

**Lemma 3.1.** *As  $T \rightarrow \infty$  we have*

$$\lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr dt = \frac{4}{\sqrt{\pi}} |\Gamma(3/4)|^2.$$

*Proof.* Using expression (2.11) we write

$$\int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr dt = \Re \left( \int_{-\infty}^{\infty} \frac{1}{T} \int_0^T G(t) \Gamma(it) e^{irt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2r} + 1} \right) dr dt \right).$$

Let  $\varepsilon > 0$  be a fixed small number and  $M > 0$  be a fixed large number. We consider the path integral

$$\int_{\gamma} G(z) \Gamma(iz) \frac{1}{T} \int_0^T e^{irz} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2r} + 1} \right) dr dz,$$

where  $\gamma$  is the contour  $\gamma = \bigcup_{i=1}^6 C_i$  with

$$\begin{aligned} C_1 &= [\epsilon, M], \\ C_2 &= \{M + iv, v \in [0, 1/2]\}, \\ C_3 &= [-M + i/2, M + i/2], \\ C_4 &= \{-M + iv, v \in [0, 1/2]\}, \\ C_5 &= [-M, -\epsilon], \\ C_6 &= \{\epsilon \cdot e^{i\theta}, \theta \in [0, \pi]\}, \end{aligned}$$

traversed counterclockwise. We write  $G(z)$  as

$$G(z) = \frac{\sqrt{2} \Gamma\left(\frac{3}{4} + \frac{iz}{2}\right) \Gamma\left(\frac{3}{4} - \frac{iz}{2}\right)}{\pi \Gamma(3/2 + iz)} \left( e^{-\frac{i\pi}{4} - \frac{\pi z}{2}} + e^{\frac{i\pi}{4} + \frac{\pi z}{2}} \right),$$

hence we see that the integrand is holomorphic inside the contour. The simple pole at  $z = 0$  is coming from  $\Gamma(iz)$ . We note that  $\text{Res}_{z=0} \Gamma(iz) = -i$ . Applying Stirling's formula and the asymptotics of the hypergeometric function (2.12) we deduce

$$\begin{aligned} \int_{C_2+C_4} G(z) \Gamma(iz) \frac{1}{T} \int_0^T e^{irz} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2r} + 1}\right) dr dz &= O(M^{-2} T^{-1}), \\ \int_{C_3} G(z) \Gamma(iz) \frac{1}{T} \int_0^T e^{irz} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2r} + 1}\right) dr dz &= O(T^{-1}). \end{aligned}$$

Further, as  $\epsilon \rightarrow 0$  we see that the term

$$\int_{C_6} G(z) \Gamma(iz) \frac{1}{T} \int_0^T e^{irz} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2r} + 1}\right) dr dz$$

converges to

$$-i\pi G(0) \frac{1}{T} \int_0^T F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{1}{e^{2r} + 1}\right) dr \text{Res}_{z=0} \Gamma(iz) = -\pi G(0)(1 + O(T^{-1})).$$

From Cauchy's Theorem we conclude

$$\begin{aligned} \int_{-M}^M G(t) \Gamma(it) \frac{1}{T} \int_0^T e^{irt} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2r} + 1}\right) dr dt &= \pi G(0)(1 + O(T^{-1})) \\ &\quad + O(M^{-2} T^{-1} + T^{-1}). \end{aligned}$$

As  $M \rightarrow \infty$  we get

$$\int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr dt = 2 \frac{\Gamma(3/4)^2}{\Gamma(3/2)} + O(T^{-1}),$$

and for  $T \rightarrow \infty$  the statement follows.  $\square$

We let  $\phi_{\mathcal{H}, \mathfrak{a}}(t)$  denote the function

$$(3.2) \quad \phi_{\mathcal{H}, \mathfrak{a}}(t) = \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z, 1/2 + it) - \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2).$$

Thus, the contribution of the cusp  $\mathfrak{a}$  in eq. (3.1) can be written in the form

$$\begin{aligned} (3.3) \quad & \frac{1}{4\pi} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2) \int_{-\infty}^{\infty} \left( \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr \right) dt \\ & + \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\mathcal{H}, \mathfrak{a}}(t) \left( \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr \right) dt. \end{aligned}$$

The second term of (3.3) can be handled using Lemma 2.2. We calculate:

$$\begin{aligned} \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\mathcal{H},\mathfrak{a}}(t) \left( \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr \right) dt &= \frac{1}{2\sqrt{2}\pi^2} \int_{-\infty}^{\infty} \phi_{\mathcal{H},\mathfrak{a}}(t) G(t) \Gamma(it) \frac{e^{itT} - 1}{iT} dt \\ &\quad + O \left( \frac{1}{T} \int_{-\infty}^{\infty} \phi_{\mathcal{H},\mathfrak{a}}(t) \frac{G(t) \Gamma(it)}{(1+|t|)(2+|t|)} dt \right). \end{aligned}$$

Since  $\phi_{\mathcal{H},\mathfrak{a}}(0) = 0$ , applying Theorems 1.7 and 2.4 we conclude the bound

$$\int_{-\infty}^{\infty} \phi_{\mathcal{H},\mathfrak{a}}(t) \left( \frac{1}{T} \int_0^T \frac{2d(f_x, t)}{x^{1/2}} dr \right) dt = O(T^{-1}).$$

Hence, as  $T \rightarrow \infty$  the contribution of the continuous spectrum converges to

$$\pi^{-3/2} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2).$$

This completes the proof of Theorem 1.5.

#### 4. $\Omega$ -RESULTS FOR THE AVERAGE ERROR TERM ON GEODESICS

In this section we give the proof of Theorem 1.9. For this reason, we mollify the average of the error term on the geodesic  $\ell$ . Let  $\psi \geq 0$  be a smooth even function compactly supported in  $[-1, 1]$ , such that  $\hat{\psi} \geq 0$  and  $\int_{-\infty}^{\infty} \psi(x) dx = 1$ . For every  $\epsilon > 0$  we also define the family of functions  $\psi_{\epsilon}(x) = \epsilon^{-1} \psi(x/\epsilon)$ . We have  $0 \leq \hat{\psi}_{\epsilon}(x) \leq 1$  and  $\hat{\psi}_{\epsilon}(0) = 1$ . As before, we study separately the contributions of the discrete and the continuous spectrum.

**4.1. The contribution of the discrete spectrum.** Let us denote by  $e(\mathcal{H}, R)$  the average of the normalized error term on the geodesic, evaluated at the parameter  $R = \log(X + U)$ , i.e.

$$e(\mathcal{H}, R) =: \int_{\mathcal{H}} \frac{e(\mathcal{H}, X; z)}{X^{1/2}} ds(z),$$

and we consider the convolution

$$(e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R) =: \int_{-\infty}^{+\infty} \psi_{\epsilon}(R - Y) e(\mathcal{H}, Y) dY.$$

In order to prove an  $\Omega$ -result for the average  $\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds$ , it suffices to prove an  $\Omega$ -result for the convolution  $(e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R)$ . Using Lemma 2.2, Stirling's asymptotic (2.7), Theorem 1.7 and the properties of  $\psi$  we calculate the contribution of the discrete spectrum in  $(e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R)$  is given by

$$\begin{aligned} &\sum_{t_j > 0} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \int_{-\infty}^{+\infty} \psi_{\epsilon}(Y - R) e^{it_j Y} dY \right) \\ &+ O \left( e^{-2R} \sum_{t_j > 0} \frac{|\hat{u}_j|^2}{t_j^2} \int_{-\infty}^{+\infty} \psi_{\epsilon}(Y - R) e^{it_j Y} dY + e^{-\sigma R} \right) \\ &= \sum_{t_j > 0} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) e^{it_j R} \right) \hat{\psi}_{\epsilon}(t_j) + O(e^{-\sigma R}). \end{aligned}$$

Let  $A > 1$ . We split the sum of the above main term for  $t_j \geq A$  and  $t_j < A$ . Using the bound

$$(4.1) \quad \hat{\psi}_{\epsilon}(t_j) = O_k((\epsilon |t_j|)^{-k})$$

for every  $k \geq 1$ , for  $t_j \geq A$  we get

$$\sum_{t_j \geq A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) e^{it_j R} \right) \hat{\psi}_{\epsilon}(t_j) = O_k(\epsilon^{-k} A^{-k}).$$

For the partial sum part of the series we use the following lemma:

**Lemma 4.1** (Dirichlet's box principle [20]). *Let  $r_1, r_2, \dots, r_n$  be  $n$  distinct real numbers and  $M > 0$ ,  $T > 1$ . Then, there is an  $R$  satisfying  $M \leq R \leq MT^n$ , such that*

$$|e^{ir_j R} - 1| < \frac{1}{T}$$

for all  $j = 1, \dots, n$ .

We apply Lemma 4.1 to the sequence  $e^{it_j R}$  and Lemma 1.7. Given  $T$  large we find an  $R$  such that  $M \ll R \ll MT^n \ll MT^{A^2}$ . The contribution of the discrete spectrum in the convoluted error term  $(e(\mathcal{H}, \cdot) * \psi_\epsilon)(R)$  takes the form

$$(4.2) \quad \sum_{t_j < A} |\hat{u}_j|^2 \Re(G(t_j)\Gamma(it_j)) \hat{\psi}_\epsilon(t_j) + O_k(T^{-1} \log A + \epsilon^{-k} A^{-k} + e^{-\sigma R}).$$

The balance  $A \log A = T$ ,  $\log M \asymp \epsilon^{-1}$ ,  $\epsilon^{-2} = A$  implies  $\log \log R \asymp \log(\epsilon^{-1})$  and for  $\epsilon \leq 1$  we get:

$$T^{-1} \log A + \epsilon^{-k} A^{-k} + e^{-\sigma R} = O(\epsilon + e^{-\sigma R}).$$

From part a) of Lemma 2.3 we conclude the sum in (4.2) is positive. On the other hand there exists one  $\tau \in (0, 1)$  such that  $\hat{\psi}(x) \geq 1/2$  for  $|x| \leq \tau$ . Since  $\hat{\psi}_\epsilon(t_j) = \hat{\psi}(\epsilon t_j)$ , we get

$$\begin{aligned} \sum_{t_j < A} \Re(G(t_j)\Gamma(it_j)) \hat{\psi}_\epsilon(t_j) |\hat{u}_j|^2 &\gg \sum_{t_j < \tau/\epsilon} \Re(G(t_j)\Gamma(it_j)) |\hat{u}_j|^2 \\ &\gg \sum_{t_j < \tau/\epsilon} t_j^{-1} |\hat{u}_j|^2. \end{aligned}$$

When  $\Gamma$  is cocompact or has sufficiently small Eisenstein periods in the sense of Definition 1.8, we have

$$\sum_{t_j < \tau/\epsilon} t_j^{-1} |\hat{u}_j|^2 \gg \log(\epsilon^{-1}) \gg \log \log R.$$

We conclude that the contribution of the discrete spectrum in  $e(\mathcal{H}, R)$  is  $\Omega_+(\log \log R)$ . This implies that if  $\Gamma$  is cocompact or has sufficiently small Eisenstein periods, the contribution of the discrete spectrum in  $\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds$  is  $\Omega_+(X^{1/2} \log \log \log X)$ . In particular, this completes the proof of Theorem 1.9 for  $\Gamma$  cocompact.

**4.2. The contribution of the continuous spectrum.** The contribution of the continuous spectrum in  $(e(\mathcal{H}, \cdot) * \psi_\epsilon)(R)$  is given by the quantity

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \Re \left( G(t)\Gamma(it) e^{iRt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}_\epsilon(t) dt.$$

Let  $\chi_{\mathcal{H}, \mathfrak{a}}(t)$  denote the function  $\chi_{\mathcal{H}, \mathfrak{a}}(t) = |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 - |\hat{E}_{\mathfrak{a}}(1/2)|^2$ . Thus the contribution of cusp  $\mathfrak{a}$  in  $(e(\mathcal{H}, \cdot) * \psi_\epsilon)(R)$  splits in

$$\begin{aligned} &\frac{|\hat{E}_{\mathfrak{a}}(1/2)|^2}{4\pi} \int_{-\infty}^{\infty} \Re \left( G(t)\Gamma(it) e^{iRt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}_\epsilon(t) dt \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \chi_{\mathcal{H}, \mathfrak{a}}(t) \Re \left( G(t)\Gamma(it) e^{iRt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}_\epsilon(t) dt. \end{aligned}$$

Let  $\gamma$  be the contour  $\gamma = \bigcup_{i=1}^6 C_i$  defined in the proof of Lemma 3.1. The function  $\psi_\epsilon(x)$  is compactly supported in the interval  $[-\epsilon, \epsilon]$ . Applying the Paley-Wiener Theorem [15, Theorem 7.4] we deduce that the holomorphic Fourier transform of  $\psi_\epsilon(x)$ :

$$\hat{\psi}_\epsilon(z) = \int_{-\infty}^{\infty} \psi_\epsilon(x) e^{-ixz} dx$$

is an entire function of type  $\epsilon$ , i.e.  $|\hat{\psi}_\epsilon(z)| \ll e^{\epsilon|z|}$ , and it is square-integrable over horizontal lines:

$$\int_{-\infty}^{\infty} |\hat{\psi}_\epsilon(v + iu)|^2 dv < \infty.$$

For fixed  $\epsilon > 0$  we have

$$\int_{-\infty}^{\infty} |\hat{\psi}_{\epsilon}(v + iu)|^2 dv = \epsilon^{-1} \int_{-\infty}^{\infty} |\hat{\psi}(v + i\epsilon u)|^2 dv$$

and since  $\int_{-\infty}^{\infty} |\hat{\psi}(v + i\epsilon u)|^2 dv$  converges uniformly to  $\int_{-\infty}^{\infty} |\hat{\psi}(v)|^2 dv$  as  $\epsilon \rightarrow 0$  we get

$$(4.3) \quad \int_{-\infty}^{\infty} |\hat{\psi}_{\epsilon}(v + i/2)|^2 dv \ll \epsilon^{-1}.$$

Consider the integral

$$\int_{\gamma} G(z) \Gamma(iz) e^{iRz} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2R} + 1}\right) \hat{\psi}_{\epsilon}(z) dz.$$

The integrand is holomorphic inside the contour. Working as in the proof of Lemma 3.1 and applying Cauchy-Schwarz inequality and bound (4.3) for the integral over  $C_3$  we deduce

$$\begin{aligned} \int_{-\infty}^{\infty} G(t) \Gamma(it) e^{iRt} F\left(-\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1}\right) \hat{\psi}_{\epsilon}(t) dt &= \pi G(0) \hat{\psi}_{\epsilon}(0) (1 + O(e^{-2R})) \\ &\quad + O(\epsilon^{-1} e^{-R/2}). \end{aligned}$$

To finish the proof of part (a) of Theorem 1.9, we notice that if

$$\int_{-T}^T |\hat{E}_{\mathbf{a}}(1/2 + it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}},$$

then the function

$$H_1(t) = \chi_{\mathcal{H}, \mathbf{a}}(t) G(t) \Gamma(it) F\left(-\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1}\right) \hat{\psi}_{\epsilon}(t)$$

is in  $L^1(\mathbb{R})$  independently of  $\epsilon$  and  $R$ . To obtain this we notice that  $\chi_{\mathcal{H}, \mathbf{a}}(t) \Gamma(it)$  remains bounded close to  $t = 0$ , we use the trivial bound  $\hat{\psi}_{\epsilon}(t) \leq 1$ , Lemma 2.2 and we estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |H_1(t)| dt &\ll \int_{-1}^1 |H_1(t)| dt + \sum_{n=0}^{\infty} 2 \int_{2^n}^{2^{n+1}} |t|^{-1} |\hat{E}_{\mathbf{a}}(1/2 + it)|^2 dt \\ (4.4) \quad &\ll \int_{-1}^1 |H_1(t)| dt + \sum_{n=0}^{\infty} 2^{-n} \int_{2^n}^{2^{n+1}} |\hat{E}_{\mathbf{a}}(1/2 + it)|^2 dt \\ &\ll \int_{-1}^1 |H_1(t)| dt + \sum_{n=0}^{\infty} \frac{1}{(n+1)^{1+\delta}} \ll 1. \end{aligned}$$

Applying the Riemann–Lebesgue Lemma we conclude that

$$(4.5) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} H_1(t) e^{iRt} dt = 0.$$

Since  $\hat{\psi}_{\epsilon}(0) = 1$  and  $\pi G(0) = 4\pi^{-1/2} |\Gamma(3/4)|^2$ , the contribution of the continuous spectrum in  $(e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R)$  takes the form

$$(4.6) \quad \frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_{\mathbf{a}} |\hat{E}_{\mathbf{a}}(1/2)|^2 + O(\epsilon^{-1} e^{-R/2}) + o(1).$$

As in the discrete spectrum (see the balance after expansion (4.2)) we choose the balance  $\epsilon^{-1} \ll \log R \ll \log \log X$ . Hence (4.6) takes the form

$$(4.7) \quad \frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_{\mathbf{a}} |\hat{E}_{\mathbf{a}}(1/2)|^2 + O(X^{-1/2} \log \log X) + o(1).$$

In particular, this completes the proof of part (a) of Theorem 1.9.

To prove part (b), we first notice that the contribution from the discrete spectrum is  $c(R) + O_k(T^{-1} \log A + \epsilon^{-k} A^{-k} + e^{-\sigma R})$ , where  $c(R) = \Omega_+(1)$  if there exists one  $\hat{u}_j \neq 0$  and  $c(R)$  vanishes otherwise. In this case, the contribution of the continuous spectrum takes the form

$$\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1} e^{-R/2}) + \epsilon^{-1} \int_{-\infty}^{\infty} H_2(t) e^{iRt} dt$$

where, using Theorem 1.7 and estimate (4.1), we deduce that the function  $H_2(t) := \epsilon H_1(t)$  is in  $L^1(\mathbb{R})$  independently of  $\epsilon$  and  $R$ . Applying the Riemann–Lebesgue Lemma, the contribution of the continuous spectrum becomes

$$\pi^{-3/2} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1} e^{-R/2}) + \epsilon^{-1} Q(R),$$

with  $Q(R) = o(1)$  as  $R \rightarrow \infty$ . We choose the balance  $\epsilon^{-2} = A$ . For  $\epsilon = \epsilon_0$  sufficiently small and fixed and letting  $R, T \rightarrow \infty$  we conclude that the convoluted normalized error  $(e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R)$  takes the form

$$c(R) + \pi^{-3/2} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + o(1),$$

The second summand is  $\Omega_+(1)$  if and only if  $\hat{E}_{\mathfrak{a}}(1/2) \neq 0$  for at least one cusp  $\mathfrak{a}$ . Part (b) now follows.

**Remark 4.2.** For part (a) of Theorem 1.9, even if  $\Gamma$  has not sufficiently small Eisenstein periods associated to  $\mathcal{H}$  but has sufficiently many cusp forms in the sense that

$$(4.8) \quad \sum_{0 < t_j < T} |\hat{u}_j|^2 \gg T,$$

we can derive the  $\Omega_+(X^{1/2} \log \log \log X)$  bound if we have a polynomial bound for the derivatives of the Eisenstein series on the critical line (see [3, Chapter 4] for details).

**4.3. An arithmetic case: the modular group.** In this subsection we concentrate to  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . The set of primitive indefinite quadratic forms  $Q(x, y) = ax^2 + bxy + cy^2$  in two variables (that means  $(a, b, c) = 1$  and  $b^2 - 4ac = d > 0$  is not a square) is in one-to-one correspondence with the set of primitive hyperbolic elements of  $\Gamma$  (see [21, p. 232]). Here we briefly describe this correspondence.

The automorphs of  $Q$  is the cyclic group  $\text{Aut}(Q) \subset \text{SL}_2(\mathbb{Z})$  which fixes  $Q$ , under the action

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

Let  $M_Q$  be a generator of  $\text{Aut}(Q)$ . Then the correspondence  $Q \rightarrow M_Q$  is bijective between indefinite integral quadratic forms in two variables and primitive hyperbolic elements of the modular group. Denote by  $\mathcal{H}_Q$  the conjugacy class of  $M_Q$  and by  $\ell_Q$  the  $M_Q$ -invariant geodesic. Define

$$r(Q, n) = \#(\{(x, y) \in \mathbb{Z}^2 : Q(x, y) = n\} / \text{Aut}(Q)),$$

and let  $\zeta(Q, s)$  denote the Epstein zeta function

$$(4.9) \quad \zeta(Q, s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s},$$

which is absolutely convergent in  $\Re(s) > 1$ . Hecke proved that the Eisenstein period  $\hat{E}_{\mathfrak{a}}(s)$  along a normalized segment of  $\ell_Q$  satisfies

$$(4.10) \quad \hat{E}_{\mathfrak{a}}(s) = \frac{d^{s/2} \Gamma^2(s/2)}{\zeta(2s) \Gamma(s)} \zeta(Q, s)$$

(see [23, eq. (9.5)]). The functional equation of the Eisenstein series implies the functional equation of the Epstein zeta function:

$$d^{(1-s)/2} \Gamma^2\left(\frac{1-s}{2}\right) \pi^{s-1} \zeta(Q, 1-s) = d^{s/2} \Gamma^2\left(\frac{s}{2}\right) \pi^{-s} \zeta(Q, s).$$

In particular, the functional equation implies the convexity bound on the critical line:

$$(4.11) \quad \zeta(Q, 1/2 + it) \ll_{\epsilon} (1 + |t|)^{1/2+\epsilon}, \quad t \in \mathbb{R}.$$

Further, for the Epstein zeta function  $\zeta(Q, 1/2 + it)$  the following subconvexity bound holds:

$$(4.12) \quad \zeta(Q, 1/2 + it) \ll_{\epsilon} (1 + |t|)^{1/3+\epsilon}$$

To prove this, write the Epstein zeta function  $\zeta(Q, s)$  is a linear combination of  $L$ -functions  $L(s, \chi)$ , where  $\chi$  runs through the class group characters of the number field  $Q(\sqrt{d})$ . If  $\chi$  is real, then  $L(s, \chi)$  factors into two Dirichlet  $L$ -functions and the bound follows from the classical convexity bound for Dirichlet  $L$ -functions. If  $\chi$  is complex then  $L(s, \chi)$  is an  $L$ -function associated to a Maass form of eigenvalue  $1/4$ , i.e. an  $L(1/2, u_j)$  (the last following by automorphic induction).

In this case we deduce that  $\Gamma$  has sufficiently small Eisenstein periods; in fact

$$(4.13) \quad \int_{-T}^T |\hat{E}_a(1/2 + it)|^2 dt \ll T^{2/3+\epsilon}$$

for every  $\epsilon > 0$ . To prove this, we use the bound  $|\zeta(1 + 2it)|^{-1} \ll (\log |t|)^7$  as  $|t| \rightarrow \infty$  and Stirling's formula, which imply

$$\frac{|\Gamma^2(1/4 + it/2)|}{|\Gamma(1/2 + it)|} \ll (1 + |t|)^{-1/2}.$$

Thus

$$\hat{E}_a(1/2 + it) \ll (1 + |t|)^{-1/2} (\log |t|)^7 \zeta(Q, 1/2 + it) \ll_{\epsilon} (1 + |t|)^{-1/6+\epsilon}$$

for every  $\epsilon > 0$ , and the bound (4.13) follows. In particular, the subconvexity bound (4.12) implies

$$\int_{\ell_Q} e(\mathcal{H}_Q, X; z) ds(z) = \Omega_+(X^{1/2} \log \log \log X).$$

## 5. POINTWISE $\Omega$ -RESULTS FOR THE ERROR TERM

In this section we prove Propositions 1.11, 1.12, and hence Theorem 1.13, where we consider pointwise  $\Omega$ -results for the error term  $e(\mathcal{H}, X; z)$ . We start with the discrete average. The arguments of the proofs follow the ideas from sections 3 and 4 (see [3, Chapter 4] for detailed proofs).

**5.1. Proof of Proposition 1.11: The discrete spectrum.** For  $K > 0$  an integer we pick equally spaced  $z_1, z_2, \dots, z_K$  points on the invariant closed geodesic  $\ell$  of  $\mathcal{H}$  with  $\rho(z_{i+1}, z_i) = \delta$ . Hence  $\delta = \mu(\ell)/K$ . For  $R = \log(X + U)$  we define the quantity

$$N_K(\mathcal{H}, R) = \frac{1}{K} \sum_{m=1}^K \frac{e(\mathcal{H}, X; z_m)}{X^{1/2}}$$

and we consider the convolution

$$(\psi_{\epsilon} * N_K(\mathcal{H}, \cdot))(R) = \int_{-\infty}^{\infty} \psi_{\epsilon}(R - Y) N_K(\mathcal{H}, Y) dY.$$

Using Lemma 2.2, the properties of  $\psi_{\epsilon}$ , Theorem 2.4 and Theorem 1.7 we conclude

$$(\psi_{\epsilon} * N(\mathcal{H}, \cdot)_K)(R) = \sum_{t_j > 0} \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^K u_j(z_m) \right) \Re(G(t_j) \Gamma(it_j) e^{it_j R}) \hat{\psi}_{\epsilon}(t_j) + O(e^{-\sigma R}).$$

For  $A > 1$ , using Stirling's formula, Theorem 2.4, Theorem 1.7 and estimate 4.1 for  $k \geq 1$  we estimate the tail of the series for  $t_j > A$  is  $O_k(\epsilon^{-k} A^{1/2-k})$ . The partial sum of the series for  $t_j \leq A$  can be handled as follows: by the definition of the period integral  $\hat{u}_j$ , as  $K \rightarrow \infty$  we get

$$\frac{\mu(\ell)}{K} \sum_{m=1}^K u_j(z_m) = \sum_{m=1}^K u_j(z_m) \delta \rightarrow \overline{u}_j$$



uniformly, for every  $j = 1, \dots, n$  (where  $n$  is such that  $t_n \leq A < t_{n+1}$ , hence  $n \asymp A^2$ ). That means for every small  $\epsilon_1 > 0$  there exists a  $K_0 = K_0(\epsilon_1) \geq 1$  such that

$$(5.1) \quad \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^K u_j(z_m) \right) = \frac{|\hat{u}_j|^2}{\mu(\ell)} + O(\epsilon_1 \hat{u}_j)$$

for every  $K \geq K_0$ . We get

$$\begin{aligned} (\psi_\epsilon * N(\mathcal{H}, \cdot)_K)(R) &= \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re(G(t_j) \Gamma(it_j) e^{it_j R}) \hat{\psi}_\epsilon(t_j) \\ &\quad + O_k \left( \epsilon_1 \sum_{t_j \leq A} \hat{u}_j \Re(G(t_j) \Gamma(it_j) e^{it_j R}) \hat{\psi}_\epsilon(t_j) + \epsilon^{-k} A^{1/2-k} + e^{-\sigma R} \right). \end{aligned}$$

Using Theorem 1.7 the  $O$ -term is bounded by  $O(\epsilon_1 A^{1/2})$ . For the main term, apply Dirichlet's principle (Lemma 4.1) to the exponentials  $e^{it_j R}$ . For every  $M$  and  $T$  we find  $M \ll R \ll MT^{A^2}$  such that

$$\begin{aligned} (\psi_\epsilon * N(\mathcal{H}, \cdot)_K)(R) &= \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re(G(t_j) \Gamma(it_j)) \hat{\psi}_\epsilon(t_j) \\ &\quad + O_k(\epsilon^{-k} A^{1/2-k} + T^{-1} \log A + \epsilon_1 A^{1/2} + e^{-\sigma R}). \end{aligned}$$

The balance  $\epsilon^{-1} = A^{1-3/(2k+2)}$ ,  $\epsilon_1 = A^{-1/2} \epsilon$  implies the  $O$ -term is  $O(T^{-1} \log A + \epsilon + e^{-\sigma R})$ . By Lemma 2.3, the coefficients of the above sum are all positive. For the function  $\psi$  we pick  $\tau \in (0, 1)$  such that  $\hat{\psi}(x) \geq 1/2$  for  $|x| \leq \tau$ . It follows that if  $\Gamma$  is cocompact or has sufficiently small Eisenstein periods we bound the above sum from below by

$$\frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re(G(t_j) \Gamma(it_j)) \hat{\psi}_\epsilon(t_j) \gg \log(\epsilon^{-1}).$$

We deduce that for every  $\epsilon > 0$  we can find a sufficiently large  $K = K(\epsilon)$  such that

$$(\psi_\epsilon * N_K(\mathcal{H}, \cdot))(R) = k(\epsilon) + O(\epsilon + e^{-\sigma R}).$$

with  $k(\epsilon) = \Omega_+(\log(\epsilon^{-1}))$ . If  $\Gamma$  is cocompact, choosing  $\epsilon = \epsilon_0$  sufficiently small and  $K = K(\epsilon_0)$  sufficiently large, for  $R, T \rightarrow \infty$  we conclude Proposition 1.11 for  $\Gamma$  cocompact.

**5.2. The continuous spectrum.** The contribution of the continuous spectrum in the convolution  $(\psi_\epsilon * N_K(\mathcal{H}, \cdot))(R)$  is given by

$$(5.2) \quad \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_{\mathfrak{a}}(1/2 + it) \left( \frac{1}{K} \sum_{m=1}^K E_{\mathfrak{a}}(z_m, 1/2 + it) \right) \cdot \Re \left( G(t) \Gamma(it) F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) e^{itR} \right) \hat{\psi}_\epsilon(t) dt.$$

For  $A > 0$ , by Theorem 1.7, asymptotic (2.8) and estimate (4.1) it follows that the contribution of  $|t| > A$  in the above integral is  $O(\epsilon^{-k} A^{1/2-k})$ . For  $|t| \leq A$  and for any small  $\epsilon_2 > 0$  we approximate the Eisenstein period integral as

$$(5.3) \quad \frac{1}{K} \sum_{m=1}^K E_{\mathfrak{a}}(z_m, 1/2 + it) = \hat{E}_{\mathfrak{a}}(1/2 - it) + O(\epsilon_2)$$

for every  $K \geq K_0$  with  $K_0 = K_0(\epsilon_2)$  sufficiently large. The contribution of the continuous spectrum (5.2) takes the form

$$(5.4) \quad \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{|t| \leq A} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \Re \left( G(t) \Gamma(it) e^{iRt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}_{\epsilon}(t) dt \\ + O_k \left( \epsilon_2 \sum_{\mathfrak{a}} \int_{|t| \leq A} \hat{E}_{\mathfrak{a}}(1/2 + it) G(t) \Gamma(it) e^{iRt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \hat{\psi}_{\epsilon}(t) dt + \epsilon^{-k} A^{1/2-k} \right).$$

By subsection 4.2 and Theorem 1.7, the first summand of (5.4) takes the form

$$(5.5) \quad \frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1} Q_1(R) + \epsilon^{-k} A^{-k}),$$

with  $Q_1(R) \rightarrow 0$  as  $R \rightarrow \infty$ . For the second summand of (5.4), we set  $\theta_{\mathcal{H}, \mathfrak{a}}(t) = \hat{E}_{\mathfrak{a}}(1/2 + it) - \hat{E}_{\mathfrak{a}}(1/2)$  and we use the contour integral method to deduce that the contribution of the continuous spectrum in  $(\psi_{\epsilon} * N_K(\mathcal{H}, \cdot))(R)$  is

$$\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O_k \left( \epsilon^{-1} Q_1(R) + \epsilon^{-k} A^{-k+1/2} + \epsilon_2 + \epsilon_2 \epsilon^{-1} e^{-R/2} + \epsilon_2 \log A \right).$$

Choosing  $\epsilon_2 = \epsilon^2$  and  $\epsilon^{-1} = A^{1-3/(2k+2)}$  as before we conclude the  $O$ -term is  $O(\epsilon^{-1} Q_1(R) + \epsilon)$ . If  $\Gamma$  has at least one  $\hat{u}_j \neq 0$  with  $\lambda_j > 1/4$  then for fixed and sufficiently small  $\epsilon$  the contribution of the discrete spectrum in  $(\psi_{\epsilon} * N_K(\mathcal{H}, \cdot))(R)$  is  $\Omega_+(1)$ . If  $\Gamma$  has at least one nonzero Eisenstein period integral then for fixed and sufficiently small  $\epsilon$  we get that the contribution of the continuous spectrum in  $(\psi_{\epsilon} * N_K(\mathcal{H}, \cdot))(R)$  is also  $\Omega_+(1)$ . This completes the proof of Proposition 1.11.

**5.3. Proof of Proposition 1.12.** In this subsection we prove Proposition 1.12, where we study the average of a normalized error term on the geodesic  $\ell$ . As we have already mentioned, this completes the proof of Theorem 1.13. We will need the following lemma for the Huber transform.

**Lemma 5.1.** *For  $y = x + \sqrt{x^2 - 1}$  we have*

$$\lim_{Y \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{Y} \int_1^Y \frac{2d(f_x, t)}{x^{1/2}} dy dt = \frac{4}{\sqrt{\pi}} |\Gamma(3/4)|^2.$$

The proof of Lemma follows similarly with that of Lemma 3.1, with the contour integration technique. We can now prove Proposition 1.12.

*Proof.* (of Proposition 1.12). Assume first that  $\Gamma$  is cocompact. We pick  $z_1, z_2, \dots, z_K$  equally spaced points on the invariant closed geodesic  $\ell$  of  $\mathcal{H}$  with  $\rho(z_{i+1}, z_i) = \delta$ . Using Lemma 2.2, Theorem 2.4 and Theorem 1.7 we conclude

$$(5.6) \quad \frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X) = \sum_{t_j > 0} \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^K u_j(z_m) \right) \Re \left( G(t_j) \Gamma(it_j) \frac{1}{Y} \int_1^Y e^{it_j r} dy \right) + O(Y^{-\sigma}),$$

For  $A > 1$ , we use Theorem 1.7 and we apply the estimate (4.1) to bound the tail of the series in (5.6) for  $t_j \geq A$  by  $O(A^{-1/2})$ . For the partial sum of the series, we approximate the period integral  $\hat{u}_j$  uniformly, for every  $j = 1, \dots, n$  (where  $n \asymp A^2$ ). For any  $\epsilon_1 > 0$  we find a  $K_0 = K_0(\epsilon_1) \geq 1$  such that for every  $K \geq K_0$ :

$$(5.7) \quad \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^K u_j(z_m) \right) = \frac{|\hat{u}_j|^2}{\mu(\ell)} + O(\epsilon_1 \hat{u}_j).$$

We get

$$\frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X) = \frac{1}{\mu(\ell)} \sum_{t_j < A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \frac{Y^{it_j}}{1 + it_j} \right) + O(Y^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).$$

For the main term, apply Dirichlet's principle (Lemma 4.1) to the exponentials  $e^{it_j R} = Y^{it_j}$ . For each  $T$  we can find  $R \ll T^{A^2}$  such that

$$\frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X) = \frac{1}{\mu(\ell)} \sum_{t_j < A} |\hat{u}_j|^2 \Re \left( \frac{G(t_j) \Gamma(it_j)}{1 + it_j} \right) + O(T^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).$$

By Theorem 1.7, as  $A \rightarrow \infty$  the sum remains bounded and, for  $\Gamma$  cocompact, there exist infinitely many  $j$ 's such that  $\hat{u}_j \neq 0$ . By Lemma 2.3, all the nonzero terms are negative. Hence, there exists an  $A_0$  such that for every  $A \geq A_0$ :

$$(5.8) \quad \left| \sum_{t_j < A} |\hat{u}_j|^2 \Re \left( \frac{G(t_j) \Gamma(it_j)}{1 + it_j} \right) \right| \gg 1.$$

For  $T, Y$  and  $A$  fixed and sufficiently large and  $\epsilon_1$  fixed and sufficiently small, we find a  $K = K_0$  fixed such that

$$\frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X) = \Omega_-(1).$$

Notice that the lower bound (5.8) holds if and only if there exists at least one nonzero  $\hat{u}_j$  with  $\lambda_j > 1/4$ .

Assume now that  $\Gamma$  is not cocompact. In this case, the contribution of the discrete spectrum in

$$\frac{1}{K} \sum_{m=1}^K M_{\mathcal{H}, z_m}(X)$$

is given by

$$(5.9) \quad \frac{1}{K} \sum_{m=1}^K \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{Y} \int_0^Y \frac{2d(f_x, t)}{x^{1/2}} dy \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z_m, 1/2 + it) dt.$$

We cut the integral for  $|t| \leq A$  and  $|t| > A$ . In the interval  $|t| \leq A$  we approximate the Eisenstein period  $\epsilon_2$ -close. Applying Lemma 5.1 and following a standard calculation, expansion (5.9) takes the form

$$\begin{aligned} & \frac{|\Gamma(3/4)|^2}{\pi^{3/2}} \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + \Re \left( \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \chi_{\mathcal{H}, \mathfrak{a}}(t) \frac{G(t) \Gamma(it)}{1 + it} Y^{it} dt \right) \\ & + O(A^{-1/2} + \epsilon_2 + Y^{-1}), \end{aligned}$$

with  $K = K(\epsilon_2, A)$ . Since for  $\Gamma$  all the Eisenstein periods  $\hat{E}_{\mathfrak{a}}(1/2)$  vanish, applying Riemann–Lebesgue Lemma for the second term the proposition follows for  $A, Y$  sufficiently large and  $\epsilon_2$  sufficiently small.  $\square$

## 6. UPPER BOUNDS ON GEODESICS

In this section, we apply the key observation arising in the spectral theory of the conjugacy problem, that is the slower divergence for the sums of period integrals of Theorem 1.7, to the error terms of both the classical problem (described in subsection 1.1) and the conjugacy class problem. In particular, for the error  $e(X; z, w)$  we prove the following average result.

**Theorem 6.1.** *Let  $\ell_0$  be a closed geodesic of  $\Gamma \backslash \mathbb{H}$  and  $e(X; z, w)$  be the error term of the classical counting problem. Then*

$$\int_{\ell_0} e(X; z, w) ds(w) = O_{\ell_0}(X^{1/2} \log X).$$

The proof of this result follows the steps of the proof for the classical pointwise bound  $O(X^{2/3})$ , sketched for instance in [14]. The standard idea here is again to approximate the kernel defined  $k(u) = \chi_{[0, (x-2)/4]}$  by appropriate step functions  $k_{\pm}(u)$  and use the observation

$$\sum_{|t_j| < T} \frac{u_j(z) \hat{u}_j}{t_j^{3/2}} \ll \log T.$$

Similarly, for the error term  $e(\mathcal{H}, X; z)$  of the conjugacy class problem we can deduce the upper bound

$$\int_{\ell_0} e(\mathcal{H}, X; z) ds(z) = O_{\ell_0}(X^{1/2} \log X).$$

Since the proof of this bound follows the same steps with that of Theorem 6.1, it is omitted.

*Proof.* (of Theorem 6.1) Assume first the cocompact case. Define the functions  $k_-(u) \leq k(u) \leq k_+(u)$  by

$$(6.1) \quad k_+(u) = \begin{cases} 1, & \text{for } u \leq \frac{X-2}{4}, \\ \frac{-4u}{Y} + \frac{X+Y-2}{Y}, & \text{for } \frac{X-2}{4} \leq u \leq \frac{X+Y-2}{4}, \\ 0, & \text{for } \frac{X+Y-2}{4} \leq u, \end{cases}$$

$$(6.2) \quad k_-(u) = \begin{cases} 1, & \text{for } u \leq \frac{X-Y-2}{4}, \\ \frac{-4u}{Y} + \frac{X-2}{Y}, & \text{for } \frac{X-Y-2}{4} \leq u \leq \frac{X-2}{4}, \\ 0, & \text{for } \frac{X-2}{4} \leq u. \end{cases}$$

We denote their Selberg/Harish-Chandra transform by  $h_{\pm}(t)$ . Using equations [2, p. 2, eq.(1.2)] we get

$$e(X; z, w) \ll \sum_{t_j \in \mathbb{R} - \{0\}} h_{\pm}(t_j) u_j(z) \overline{u_j(w)} + O\left((Y + X^{1/2}) \sum_{t_j \notin \mathbb{R}} u_j(z) \overline{u_j(w)}\right),$$

hence

$$\int_{\ell_0} e(X; z, w) ds(w) \ll \sum_{t_j \in \mathbb{R} - \{0\}} h_{\pm}(t_j) u_j(z) \hat{u}_j + O(Y + X^{1/2}).$$

Using estimates [14, p. 173, eq. (12.9)] we conclude

$$(6.3) \quad \sum_{t_j \in \mathbb{R} - \{0\}} h_{\pm}(t_j) u_j(z) \hat{u}_j \ll X^{1/2} \sum_{t_j} |t_j|^{-5/2} \min\{|t_j|, XY^{-1}\} u_j(z) \hat{u}_j.$$

Applying Cauchy-Schwarz inequality, local Weyl's laws for the Maass forms  $u_j(z)$  and Theorem 1.7 for the periods  $\hat{u}_j$ , we deduce that (6.3) bounded by

$$X^{1/2} \sum_{t_j \leq X/Y} |t_j|^{-3/2} u_j(z) \hat{u}_j + X^{1/2} \sum_{t_j > X/Y} |t_j|^{-5/2} \frac{X}{Y} u_j(z) \hat{u}_j \ll X^{1/2} \log(X/Y) + X^{1/2}.$$

We conclude

$$\int_{\ell_0} e(X; z, w) ds(w) \ll X^{1/2} \log(X/Y) + Y + X^{1/2}$$

and the statement follows for  $Y = X^{1/2}$ . For the cofinite case, the result follows similarly, using the relevant bounds for the Eisenstein series and their period integrals.  $\square$

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SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL, BS8 1TW, UK, AND DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET, LONDON WC1E 6BT  
*E-mail address:* d.chatzacos.12@ucl.ac.uk